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Least squares solutions to $AX = B$ for bisymmetric matrices under a central principal submatrix constraint and the optimal approximation ☆

Lijun Zhao *, Xiyan Hu, Lei Zhang

College of Mathematics and Econometrics, Hunan University, Changsha 410082, PR China

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Abstract

A matrix $A \in R^{n\times n}$ is called a bisymmetric matrix if its elements $a_{i,j}$ satisfy the properties $a_{i,j} = a_{j,i}$ and $a_{i,j} = a_{n-j+1,n-i+1}$ for $1 \leqslant i, j \leqslant n$. This paper considers least squares solutions to the matrix equation $AX = B$ for A under a central principal submatrix constraint and the optimal approximation. A central principal submatrix is a submatrix obtained by deleting the same number of rows and columns in edges of a given matrix. We first discuss the specified structure of bisymmetric matrices and their central principal submatrices. Then we give some necessary and sufficient conditions for the solvability of the least squares problem, and derive the general representation of the solutions. Moreover, we also obtain the expression of the solution to the corresponding optimal approximation problem.

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Keywords: Bisymmetric matrix; Central principal submatrix; Least squares problem; Optimal approximation

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* Corresponding author.
E-mail address: zhaolijun82@yahoo.com.cn (L. Zhao).

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1. Introduction

We first introduce some notations to be used. Let $R^{n\times m}$ denote the set of all $n\times m$ real matrices; $\mathrm{SR}^{n\times n}$, $\mathrm{ASR}^{n\times n}$ and $\mathrm{OR}^{n\times n}$ be the sets of all $n\times n$ symmetric, antisymmetric and orthogonal matrices, respectively; A^{T}, $\mathrm{rank}(A)$ and A^{+} represent the transpose, rank and Moore–Penrose generalized inverse of matrix A, respectively; I_n denote the identity matrix of order n; 0 be a zero matrix or vector of size implied by context. We use $\langle A, B\rangle = \mathrm{trace}(B^{\mathrm{T}}A)$ to define the inner product of matrices A and B in $R^{n\times m}$. Then $R^{n\times m}$ is a Hilbert inner product space. The norm of a matrix generated by the inner product is the Frobenius norm $\|\cdot\|$, that is $\|A\| = \sqrt{\langle A, A\rangle} = (\mathrm{trace}(A^{\mathrm{T}}A))^{\frac{1}{2}}$. For $A=(a_{i,j})$, $B=(b_{i,j})\in R^{n\times m}$, $A*B=(a_{i,j}b_{i,j})$ represents the Hadamard product of A and B.

Definition 1. A matrix $A\in R^{n\times n}$ is called a bisymmetric matrix if its elements $a_{i,j}$ satisfy the properties

$$a_{i,j}=a_{j,i} \quad \text{and} \quad a_{i,j}=a_{n-j+1,n-i+1}, \quad \text{for } 1\leqslant i,j\leqslant n.$$

The set of all $n\times n$ bisymmetric matrices is denoted by $\mathrm{BSR}^{n\times n}$.

Bisymmetric matrices have been widely discussed since 1939, which are very useful in engineering and statistics [10], communication theory [11], and numerical analysis theory [12], and others. In fact, the symmetric Toeplitz matrices and centrosymmetric Hankel matrices are two useful examples of bisymmetric matrices. For important results on the discussions of bisymmetric matrices, we refer the reader to [3,6,7] and the references contained therein. In particular, Peng et al. [3] considered a linear constrained problem of bisymmetric matrices with a leading principal submatrix constraint. Because of the specified structure of bisymmetric matrices, it is unfit for discussing bisymmetric matrices under their leading principal submatrices (i.e. the principal submatrices lie in the left-top of a given matrix) constraint, for they destroy the special symmetric of bisymmetric matrices. Therefore, we present a concept of central principal submatrix, which was originally proposed by Yin [8]. The definition is as follows.

Definition 2. Given $A\in R^{n\times n}$, if $n-k$ is even, then a k-square central principal submatrix of A, denoted as $A_c(k)$, is a k-square submatrix obtained by deleting the first and last $(n-k)/2$ rows and columns of A, that is

$$A_c(k)=(0, I_k, 0)A(0, I_k, 0)^{\mathrm{T}}, \quad 0\in R^{k\times\frac{n-k}{2}}.$$

It is intuitive and obvious that a matrix of odd (even) order only has central principal submatrices of odd (even) order.

Throughout this paper, we denote by $A[k]$ the right-bottom $k\times k$ principal submatrix of $A\in R^{n\times n}$, that is

$$A[k]=(a_{i,j})_{i,j=n-k+1}^{n}=(0,\ I_k)A(0,\ I_k)^{\mathrm{T}}.$$

The problem of finding solutions to a matrix equation under a submatrix constraint comes from a practical subsystem expansion problem. Therefore, researchers have great interest in studying a variety of problems under submatrices constraint of late years. For example, Deift and Nanda [1] discussed an inverse eigenvalue problem of a tridiagonal matrix under a submatrix constraint; Peng and Hu [2] considered an inverse eigenpair problem of a Jacobi matrix under a leading principal submatrix constraint; Peng et al. [3] studied a linear constrained problem of bisymmetric matrices with a leading principal submatrix constraint; Gong et al. [4] discussed a least squares problem of antisymmetric matrices under a leading principal submatrix constraint. In particular, constrained or unconstrained least squares problems arise in a variety of applications, such as, control theory and the inverse Sturm–Liouville problems [13], inverse problems of vibration theory [9], and multidimensional approximation [14]. The least squares problem of bisymmetric matrices under a central principal submatrix constraint is presented in light of the special symmetric structure of bisymmetric matrices, thus this problem has practical applications. Furthermore, it extends and develops the results in [3]. However, it has not been considered yet. In this paper, we will discuss this problem and its optimal approximation. They are as follows.

Problem I. Given $X, B \in R^{n\times m}$, $A_0 \in \mathrm{BSR}^{k\times k}$, find $A \in \mathrm{BSR}^{n\times n}$ such that

$$\|AX - B\| = \min \quad \text{and} \quad A_c(k) = A_0.$$

Problem II. Given $\widetilde{A} \in R^{n\times n}$, find a matrix $A^* \in S_A$ such that

$$\|A^* - \widetilde{A}\| = \min_{\forall A \in S_A} \|A - \widetilde{A}\|,$$

where S_A is the solution set of Problem I.

The paper is organized as follows: First, we discuss the special properties and structure of bisymmetric matrices and their central principal submatrices. Depending on these, we convert the least squares problem of a bisymmetric matrix under a central principal submatrix constraint to two least squares problems of half-sized independent symmetric matrices under themselves right-bottom principal submatrices constraint trickily. This simplifies and is crucial to solve the problem involved, and is a special feature of this paper. Next, we derive the solvability conditions of Problem I and its general solutions' expression. Moreover, we also consider the case of $\|AX - B\| = \min = 0$, which is $AX = B$, and obtain the sufficient and necessary conditions of it and its general solutions' expression. Finally, we prove that Problem II has a unique solution and give the expression of it.

2. The properties of bisymmetric matrices and their central principal submatrices

Denote by e_i the ith ($i = 1, 2, \ldots, n$) column of I_n, $S_n = (e_n, e_{n-1}, \ldots, e_1)$, then

$$S_n = S_n^{\mathrm{T}}, \quad S_n S_n^{\mathrm{T}} = I_n.$$

Let $r = [n/2]$, where $[n/2]$ is the maximum integer which is not greater than $n/2$. Define D_n as

$$D_n = \frac{1}{\sqrt{2}}\begin{pmatrix} I_r & I_r \\ S_r & -S_r \end{pmatrix} (n = 2r), \quad D_n = \frac{1}{\sqrt{2}}\begin{pmatrix} I_r & 0 & I_r \\ 0 & \sqrt{2} & 0 \\ S_r & 0 & -S_r \end{pmatrix} (n = 2r + 1),$$

then it is easy verified that the above matrices D_n are orthogonal matrices.

Lemma 1 [3]. $A \in \mathrm{BSR}^{2r\times 2r}$ *if and only if there exist* $M, N \in \mathrm{SR}^{r\times r}$ *such that*

$$A = \begin{pmatrix} M & NS_r \\ S_r N & S_r M S_r \end{pmatrix} = D_{2r}\begin{pmatrix} M + N & 0 \\ 0 & M - N \end{pmatrix} D_{2r}^{\mathrm{T}}. \tag{1}$$

$A \in \mathrm{BSR}^{(2r+1)\times(2r+1)}$ if and only if there exist $M, N \in \mathrm{SR}^{r\times r}$, $u \in R^r$ and $\alpha \in R$ such that

$$A = \begin{pmatrix} M & u & NS_r \\ u^{\mathrm{T}} & \alpha & u^{\mathrm{T}}S_r \\ S_rN & S_ru & S_rMS_r \end{pmatrix} = D_{2r+1}\begin{pmatrix} M+N & \sqrt{2}u & 0 \\ \sqrt{2}u^{\mathrm{T}} & \alpha & 0 \\ 0 & 0 & M-N \end{pmatrix} D_{2r+1}^{\mathrm{T}}. \tag{2}$$

Furthermore, when $n = 2r$, let

$$A_{11} = M + N, \quad A_{22} = M - N, \tag{3}$$

and when $n = 2r + 1$, let

$$A_{11} = \begin{pmatrix} M+N & \sqrt{2}u \\ \sqrt{2}u^{\mathrm{T}} & \alpha \end{pmatrix}, \quad A_{22} = M - N, \tag{4}$$

then $A \in \mathrm{BSR}^{n\times n}$ if and only if there exist $A_{11} \in \mathrm{SR}^{(n-r)\times(n-r)}$ and $A_{22} \in \mathrm{SR}^{r\times r}$, whether n is odd or even, such that

$$A = D_n \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D_n^{\mathrm{T}}. \tag{5}$$

Now we give the special properties of a central principal submatrix of a bisymmetric matrix, that is the submatrix having the same symmetric properties and structure as the given bisymmetric matrix. Hence they have similar expressions, which is crucial to solve the least squares problem studied in, for it provides a reasoned way to convert the least squares problem to two least squares problems of half-sized independent symmetric matrices under themselves right-bottom principal submatrices constraint. Here, we always assume $t = [k/2]$.

Lemma 2. *Let $A \in \mathrm{BSR}^{n\times n}$ have the form as (5). Then the k-square central principal submatrix of A can be expressed as*

$$A_c(k) = D_k \begin{pmatrix} A_{11}[k-t] & 0 \\ 0 & A_{22}[t] \end{pmatrix} D_k^{\mathrm{T}}. \tag{6}$$

Proof. When $n = 2r$, from (1) and the properties of central principal submatrices, which is the matrix of even order only having central principal submatrices of even order, we have $k = 2t$, and

$$A_c(k) = \begin{pmatrix} M[t] & N[t]S_t \\ S_tN[t] & S_tM[t]S_t \end{pmatrix}.$$

Thus,

$$\begin{aligned} D_k^{\mathrm{T}}A_c(k)D_k &= \frac{1}{\sqrt{2}}\begin{pmatrix} I_t & S_t \\ I_t & -S_t \end{pmatrix} \cdot \begin{pmatrix} M[t] & N[t]S_t \\ S_tN[t] & S_tM[t]S_t \end{pmatrix} \cdot \frac{1}{\sqrt{2}}\begin{pmatrix} I_t & I_t \\ S_t & -S_t \end{pmatrix} \\ &= \begin{pmatrix} M[t]+N[t] & 0 \\ 0 & M[t]-N[t] \end{pmatrix}, \end{aligned}$$

and (3) implies $M[t] + N[t] = A_{11}[t]$ and $M[t] - N[t] = A_{22}[t]$. It says that the k-square central principal submatrix of A may be expressed as, when $n = 2r$,

$$A_c(k) = D_k \begin{pmatrix} A_{11}[t] & 0 \\ 0 & A_{22}[t] \end{pmatrix} D_k^{\mathrm{T}}. \tag{7}$$

When $n = 2r + 1$, from (2) and the properties of central principal submatrices, which is the matrix of odd order only having central principal submatrices of odd order, we have $k = 2t + 1$, and

$$A_c(k) = \begin{pmatrix} M[t] & u_t & N[t]S_t \\ u_t^{\mathrm{T}} & \alpha & u_t^{\mathrm{T}} S_t \\ S_t N[t] & S_t u_t & S_t M[t] S_t \end{pmatrix}, \quad \text{where } u_t = (0, I_t)u.$$

Hence,

$$\begin{aligned} D_k^{\mathrm{T}} A_c(k) D_k &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} I_t & 0 & S_t \\ 0 & \sqrt{2} & 0 \\ I_t & 0 & -S_t \end{pmatrix} \begin{pmatrix} M[t] & u_t & N[t]S_t \\ u_t^{\mathrm{T}} & \alpha & u_t^{\mathrm{T}} S_t \\ S_t N[t] & S_t u_t & S_t M[t] S_t \end{pmatrix} \tfrac{1}{\sqrt{2}} \begin{pmatrix} I_t & 0 & I_t \\ 0 & \sqrt{2} & 0 \\ S_t & 0 & -S_t \end{pmatrix} \\ &= \begin{pmatrix} M[t] + N[t] & \sqrt{2} u_t & 0 \\ \sqrt{2} u_t^{\mathrm{T}} & \alpha & 0 \\ 0 & 0 & M[t] - N[t] \end{pmatrix}, \end{aligned}$$

and (4) implies $\begin{pmatrix} M[t]+N[t] & \sqrt{2}u_t \\ \sqrt{2}u_t^{\mathrm{T}} & \alpha \end{pmatrix} = A_{11}[t+1]$ and $M[t] - N[t] = A_{22}[t]$. It means that the k-square central principal submatrix of A may be written as, when $n = 2r + 1$,

$$A_c(k) = D_k \begin{pmatrix} A_{11}[t+1] & 0 \\ 0 & A_{22}[t] \end{pmatrix} D_k^{\mathrm{T}}. \tag{8}$$

Combining (7) and (8), we obtain that the k-square central principal submatrix of A has the form as (6). $\square$

It is easy to verify the following lemma from Lemma 2.

Lemma 3. *Suppose $A \in \mathrm{BSR}^{n \times n}$ has the form as (5). Partition $A_0 \in \mathrm{BSR}^{k \times k}$ as*

$$A_0 = D_k \begin{pmatrix} A_{10} & 0 \\ 0 & A_{20} \end{pmatrix} D_k^{\mathrm{T}}, \quad A_{10} \in \mathrm{SR}^{(k-t)\times(k-t)}, \quad A_{20} \in \mathrm{SR}^{t \times t}, \tag{9}$$

then A_0 is a central principal submatrix of A if and only if $A_{10} = A_{11}[k-t]$ and $A_{20} = A_{22}[t]$.

3. General expression of the solutions to Problem I

Utilizing that a bisymmetric matrix and its central principal submatrices have the same expression forms, which have been discussed in previous section, we convert Problem I to two least squares problems of half-sized independent symmetric matrices under themselves right-bottom principal submatrices constraint trickily, which is a special feature of this paper. Then we solve the least squares problem of bisymmetric matrices under a central principal submatrix constraint completely. In this section, we obtain the necessary and sufficient conditions for the existence of and expression for the general solutions to Problem I. First, we introduce some useful lemmas.

Let the Singular Value Decomposition (SVD) of $X \in R^{n \times m}$ be

$$X = U \begin{pmatrix} \Sigma & 0 \\ 0 & 0 \end{pmatrix} V^{\mathrm{T}} = U_1' \Sigma V_1'^{\mathrm{T}}, \tag{10}$$

where $U = (U_1', U_2') \in \mathrm{OR}^{n \times n}$, $V = (V_1', V_2') \in \mathrm{OR}^{m \times m}$, $s = \mathrm{rank}(X)$, $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_s)$, $\sigma_i > 0 (i = 1, \ldots, s)$, $U_1' \in R^{n \times s}$ and $V_1' \in R^{m \times s}$. Define

$$\Phi = (\phi_{i,j}) \in R^{s \times s}, \quad \phi_{i,j} = \frac{1}{\sigma_i^2 + \sigma_j^2}, \quad 1 \leqslant i, j \leqslant s. \tag{11}$$

Lemma 4 [5]. *Given $X, B \in R^{n\times m}$. Let the SVD of X be* (10) *and Φ be* (11). *Denote $S_1 \equiv \{A \in \mathrm{SR}^{n\times n} | f_1(A) = \|AX - B\| = \min\}$, then the elements of S_1 may be expressed as*

$$A = U\begin{pmatrix} \Phi * ({U'_1}^{\mathrm{T}} B V'_1 \Sigma + \Sigma {V'}_1^{\mathrm{T}} B^{\mathrm{T}} U'_1) & \Sigma^{-1} {V'}_1^{\mathrm{T}} B^{\mathrm{T}} U'_2 \\ {U'}_2^{\mathrm{T}} B V'_1 \Sigma^{-1} & G \end{pmatrix} U^{\mathrm{T}}, \quad \forall G \in \mathrm{SR}^{(n-s)\times(n-s)}.$$

In particular, $f_1(A) = 0$ has solutions in $\mathrm{SR}^{n\times n}$ if and only if $X^{\mathrm{T}}B = B^{\mathrm{T}}X$ and $BX^+X = B$, and the general solutions have the following form

$$A = BX^+ + (BX^+)^{\mathrm{T}}(I - XX^+) + U'_2 G {U'}_2^{\mathrm{T}}, \quad \forall G \in \mathrm{SR}^{(n-s)\times(n-s)}.$$

Lemma 5 [9]. *Suppose the SVD of $X \in R^{n\times m}$ is* (10), *then matrix equation $X^{\mathrm{T}}CX = K$ has solutions in $\mathrm{SR}^{n\times n}$ if and only if $K^{\mathrm{T}} = K$ and $KX^+X = K$. Moreover, the general solutions can be written as*

$$C = U\begin{pmatrix} \Sigma^{-1} {V'}_1^{\mathrm{T}} K V'_1 \Sigma^{-1} & C_{12} \\ C_{12}^{\mathrm{T}} & C_{22} \end{pmatrix} U^{\mathrm{T}}, \quad \forall C_{12} \in R^{s\times(n-s)}, \ \forall C_{22} \in \mathrm{SR}^{(n-s)\times(n-s)}.$$

Theorem 1. *Given $X, B \in R^{n\times m}$ and $A_0 \in \mathrm{BSR}^{k\times k}$. Partition A_0 as* (9), *and $D_n^{\mathrm{T}}X$, $D_n^{\mathrm{T}}B$ as follows*

$$D_n^{\mathrm{T}}X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad D_n^{\mathrm{T}}B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \quad X_1, B_1 \in R^{(n-r)\times m}, \ X_2, B_2 \in R^{r\times m}.$$

Assume the SVD's of X_1 and X_2 are, respectively,

$$X_1 = U_1\begin{pmatrix} \Delta_1 & 0 \\ 0 & 0 \end{pmatrix} V_1^{\mathrm{T}}, \quad X_2 = U_2\begin{pmatrix} \Delta_2 & 0 \\ 0 & 0 \end{pmatrix} V_2^{\mathrm{T}},$$

where $U_1 = (U_{11}, U_{12}) \in \mathrm{OR}^{(n-r)\times(n-r)}$, $V_1 = (V_{11}, V_{12}) \in \mathrm{OR}^{m\times m}$, $s_1 = \mathrm{rank}(X_1)$, $\Delta_1 = \mathrm{diag}(\delta_1^{(1)}, \ldots, \delta_{s_1}^{(1)})$, $\delta_i^{(1)} > 0 (i = 1, \ldots, s_1)$, $U_{11} \in R^{(n-r)\times s_1}$, $V_{11} \in R^{m\times s_1}$; $U_2 = (U_{21}, U_{22}) \in \mathrm{OR}^{r\times r}$, $V_2 = (V_{21}, V_{22}) \in \mathrm{OR}^{m\times m}$, $s_2 = \mathrm{rank}(X_2)$, $\Delta_2 = \mathrm{diag}(\delta_1^{(2)}, \ldots, \delta_{s_2}^{(2)})$, $\delta_i^{(2)} > 0 (i = 1, \ldots, s_2)$, $U_{21} \in R^{r\times s_2}$, $V_{21} \in R^{m\times s_2}$. *Define*

$$\Phi_1 = \left(\phi_{i,j}^{(1)}\right) \in R^{s_1\times s_1}, \quad \phi_{i,j}^{(1)} = \frac{1}{(\delta_i^{(1)})^2 + (\delta_j^{(1)})^2}, \ 1 \leqslant i, j \leqslant s_1,$$

$$\Phi_2 = \left(\phi_{i,j}^{(2)}\right) \in R^{s_2\times s_2}, \quad \phi_{i,j}^{(2)} = \frac{1}{(\delta_i^{(2)})^2 + (\delta_j^{(2)})^2}, \ 1 \leqslant i, j \leqslant s_2,$$

and

$$H_1 = U_{12}^{\mathrm{T}}\begin{pmatrix} 0 \\ I_{k-t} \end{pmatrix}, \quad H_2 = U_{22}^{\mathrm{T}}\begin{pmatrix} 0 \\ I_t \end{pmatrix}. \tag{12}$$

Let the SVD's of H_1 and H_2 be, respectively,

$$H_1 = W_1\begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} Q_1^{\mathrm{T}}, \quad H_2 = W_2\begin{pmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{pmatrix} Q_2^{\mathrm{T}},$$

where $W_1 = (W_{11}, W_{12}) \in \mathrm{OR}^{(n-r-s_1)\times(n-r-s_1)}$, $Q_1 = (Q_{11}, Q_{12}) \in \mathrm{OR}^{(k-t)\times(k-t)}$, $q_1 = \mathrm{rank}(H_1)$, $\Sigma_1 = \mathrm{diag}(\sigma_1^{(1)}, \ldots, \sigma_{q_1}^{(1)})$, $\sigma_i^{(1)} > 0 (i = 1, \ldots, q_1)$, $W_{11} \in R^{(n-r-s_1)\times q_1}$, $Q_{11} \in R^{(k-t)\times q_1}$; $W_2 = (W_{21}, W_{22}) \in \mathrm{OR}^{(r-s_2)\times(r-s_2)}$, $Q_2 = (Q_{21}, Q_{22}) \in \mathrm{OR}^{t\times t}$, $q_2 = \mathrm{rank}(H_2)$, $\Sigma_2 = \mathrm{diag}(\sigma_1^{(2)}, \ldots, \sigma_{q_2}^{(2)})$, $\sigma_i^{(2)} > 0 (i = 1, \ldots, q_2)$, $W_{21} \in R^{(r-s_2)\times q_2}$, $Q_{21} \in R^{t\times q_2}$. *Set*

$$Y_{10} = U_1 \begin{pmatrix} \Phi_1 * (U_{11}^{\mathrm{T}} B_1 V_{11} \Delta_1 + \Delta_1 V_{11}^{\mathrm{T}} B_1^{\mathrm{T}} U_{11}) & \Delta_1^{-1} V_{11}^{\mathrm{T}} B_1^{\mathrm{T}} U_{12} \\ U_{12}^{\mathrm{T}} B_1 V_{11} \Delta_1^{-1} & 0 \end{pmatrix} U_1^{\mathrm{T}},$$

$$Y_{20} = U_2 \begin{pmatrix} \Phi_2 * (U_{21}^{\mathrm{T}} B_2 V_{21} \Delta_2 + \Delta_2 V_{21}^{\mathrm{T}} B_2^{\mathrm{T}} U_{21}) & \Delta_2^{-1} V_{21}^{\mathrm{T}} B_2^{\mathrm{T}} U_{22} \\ U_{22}^{\mathrm{T}} B_2 V_{21} \Delta_2^{-1} & 0 \end{pmatrix} U_2^{\mathrm{T}}, \tag{13}$$

$$K_1 = A_{10} - (0, I_{k-t}) Y_{10} (0, I_{k-t})^{\mathrm{T}}, \quad K_2 = A_{20} - (0, I_t) Y_{20} (0, I_t)^{\mathrm{T}}.$$

Then Problem I is solvable if and only if

$$K_1 H_1^+ H_1 = K_1, \quad K_2 H_2^+ H_2 = K_2. \tag{14}$$

Furthermore, every element A in the solution set S_A may be written as

$$A = D_n \begin{pmatrix} Y_{10} + U_{12} G_1 U_{12}^{\mathrm{T}} & 0 \\ 0 & Y_{20} + U_{22} G_2 U_{22}^{\mathrm{T}} \end{pmatrix} D_n^{\mathrm{T}}, \tag{15}$$

where

$$G_1 = W_1 \begin{pmatrix} \Sigma_1^{-1} Q_{11}^{\mathrm{T}} K_1 Q_{11} \Sigma_1^{-1} & E_{12} \\ E_{12}^{\mathrm{T}} & E_{22} \end{pmatrix} W_1^{\mathrm{T}}, \quad G_2 = W_2 \begin{pmatrix} \Sigma_2^{-1} Q_{21}^{\mathrm{T}} K_2 Q_{21} \Sigma_2^{-1} & F_{12} \\ F_{12}^{\mathrm{T}} & F_{22} \end{pmatrix} W_2^{\mathrm{T}}, \tag{16}$$

for $E_{22} \in \mathrm{SR}^{(n-r-s_1-q_1)\times(n-r-s_1-q_1)}$, $E_{12} \in R^{q_1\times(n-r-s_1-q_1)}$, $F_{22} \in \mathrm{SR}^{(r-s_2-q_2)\times(r-s_2-q_2)}$, $F_{12} \in R^{q_2\times(r-s_2-q_2)}$ *are arbitrary.*

Proof. By Lemmas 1 and 3, Problem I is equivalent to the existence of $A_{11} \in \mathrm{SR}^{(n-r)\times(n-r)}$ and $A_{22} \in \mathrm{SR}^{r\times r}$ such that

$$A = D_n \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix} D_n^{\mathrm{T}},$$

where A_{11} and A_{22} must satisfy

$$\|A_{11} X_1 - B_1\| = \min, \quad \|A_{22} X_2 - B_2\| = \min, \tag{17}$$

$$A_{10} = A_{11}[k-t] = (0, I_{k-t}) A_{11} (0, I_{k-t})^{\mathrm{T}}, \quad A_{20} = A_{22}[t] = (0, I_t) A_{22} (0, I_t)^{\mathrm{T}}. \tag{18}$$

From Lemma 4, (17) holds if and only if A_{11} and A_{22} are

$$A_{11} = Y_{10} + U_{12} G_1 U_{12}^{\mathrm{T}}, \quad A_{22} = Y_{20} + U_{22} G_2 U_{22}^{\mathrm{T}}, \tag{19}$$

where $G_1 \in \mathrm{SR}^{(n-r-s_1)\times(n-r-s_1)}$ and $G_2 \in \mathrm{SR}^{(r-s_2)\times(r-s_2)}$ are arbitrary matrices. Substituting (19) into (18), and noticing (12) and (13), the definitions of H_1, H_2, K_1 and K_2, then G_1 and G_2 must satisfy

$$H_1^{\mathrm{T}} G_1 H_1 = K_1, \quad H_2^{\mathrm{T}} G_2 H_2 = K_2. \tag{20}$$

Lemma 5 implies that (20) holds if and only if

$$K_1^{\mathrm{T}} = K_1, \quad K_2^{\mathrm{T}} = K_2, \quad K_1 H_1^+ H_1 = K_1, \quad K_2 H_2^+ H_2 = K_2. \tag{21}$$

As we know from (9) and (13) that K_1 and K_2 are both symmetric matrices. Hence, (21) is equivalent to (14), and G_1, G_2 can be expressed as (16), and the general solutions to Problem I may be written as (15). $\square$

We can obtain the following corollary proved in a similar way as Theorem 1.

Corollary 1. *Replace Y_{10} and Y_{20} in (13) with $Y_{10} = B_1X_1^+ + (B_1X_1^+)^{\mathrm{T}}(I - X_1X_1^+)$ and $Y_{20} = B_2X_2^+ + (B_2X_2^+)^{\mathrm{T}}(I - X_2X_2^+)$. The other symbols are the same as in Theorem 1. Then there exist solutions $A \in \mathrm{BSR}^{n\times n}$ such that $AX = B$ and $A_c(k) = A_0$ if and only if*

$$X_1^{\mathrm{T}}B_1 = B_1^{\mathrm{T}}X_1, \quad B_1X_1^+X_1 = B_1, \quad X_2^{\mathrm{T}}B_2 = B_2^{\mathrm{T}}X_2, \quad B_2X_2^+X_2 = B_2,$$
$$K_1H_1^+H_1 = K_1, \quad K_2H_2^+H_2 = K_2.$$

Furthermore, the general solutions also have the form as (15).

4. The solution to Problem II

When the solution set of Problem I is nonempty, it is easy to verify that S_A is a closed convex set, therefore there exists a unique solution A^* to Problem II. Now we give the expression of A^*.

Theorem 2. *Given $\widetilde{A} \in R^{n\times n}$, $X, B \in R^{n\times m}$ and $A_0 \in \mathrm{BSR}^{k\times k}$. Denote*

$$D_n^{\mathrm{T}}\frac{\widetilde{A} + \widetilde{A}^{\mathrm{T}}}{2}D_n = \begin{pmatrix} \widetilde{A}_{11} & \widetilde{A}_{12} \\ \widetilde{A}_{12}^{\mathrm{T}} & \widetilde{A}_{22} \end{pmatrix}, \quad \textit{where } \widetilde{A}_{11} \in \mathrm{SR}^{(n-r)\times(n-r)}, \widetilde{A}_{22} \in \mathrm{SR}^{r\times r}.$$

If Problem I is solvable, then Problem II has a unique solution A^*, which can be written as

$$A^* = D_n \begin{pmatrix} Y_{10} + U_{12}G_1U_{12}^{\mathrm{T}} & 0 \\ 0 & Y_{20} + U_{22}G_2U_{22}^{\mathrm{T}} \end{pmatrix} D_n^{\mathrm{T}}, \tag{22}$$

where

$$G_1 = W_1 \begin{pmatrix} \Sigma_1^{-1}Q_{11}^{\mathrm{T}}K_1Q_{11}\Sigma_1^{-1} & E_{12} \\ E_{12}^{\mathrm{T}} & E_{22} \end{pmatrix} W_1^{\mathrm{T}},$$

$$G_2 = W_2 \begin{pmatrix} \Sigma_2^{-1}Q_{21}^{\mathrm{T}}K_2Q_{21}\Sigma_2^{-1} & F_{12} \\ F_{12}^{\mathrm{T}} & F_{22} \end{pmatrix} W_2^{\mathrm{T}},$$

for $E_{12} = W_{11}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{12}$, $E_{22} = W_{12}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{12}$, $F_{12} = W_{21}^{\mathrm{T}}U_{22}^{\mathrm{T}}\widetilde{A}_{22}U_{22}W_{22}$, $F_{22} = W_{22}^{\mathrm{T}}U_{22}^{\mathrm{T}}\widetilde{A}_{22}U_{22}W_{22}$.

Proof. Suppose A is an arbitrary solution to Problem I, then by (15), we have

$$\begin{aligned}
&\|A - \widetilde{A}\|^2 \\
&\quad = \left\| D_n \begin{pmatrix} Y_{10} + U_{12}G_1U_{12}^{\mathrm{T}} & 0 \\ 0 & Y_{20} + U_{22}G_2U_{22}^{\mathrm{T}} \end{pmatrix} D_n^{\mathrm{T}} \right. \\
&\qquad \left. - \frac{1}{2}(\widetilde{A} + \widetilde{A}^{\mathrm{T}}) \right\|^2 + \left\| \frac{1}{2}(\widetilde{A} - \widetilde{A}^{\mathrm{T}}) \right\|^2 \\
&\quad = \left\| Y_{10} + U_{12}G_1U_{12}^{\mathrm{T}} - \widetilde{A}_{11} \right\|^2 + \left\| Y_{20} + U_{22}G_2U_{22}^{\mathrm{T}} - \widetilde{A}_{22} \right\|^2 \\
&\qquad + 2\left\|\widetilde{A}_{12}\right\|^2 + \left\| \frac{1}{2}(\widetilde{A} - \widetilde{A}^{\mathrm{T}}) \right\|^2.
\end{aligned}$$

Hence $\|A - \widetilde{A}\| = \min\limits_{A\in S_A}$ is equivalent to

$$\begin{aligned}
&\|Y_{10} + U_{12}G_1U_{12}^{\mathrm{T}} - \widetilde{A}_{11}\| = \min_{G_1\in \mathrm{SR}^{(n-r-s_1)\times(n-r-s_1)}}, \\
&\|Y_{20} + U_{22}G_2U_{22}^{\mathrm{T}} - \widetilde{A}_{22}\| = \min_{G_2\in \mathrm{SR}^{(r-s_2)\times(r-s_2)}}.
\end{aligned} \tag{23}$$

Utilizing $U_{12}^{\mathrm{T}}Y_{10}U_{12}=0$ and $U_{22}^{\mathrm{T}}Y_{20}U_{22}=0$ from (13), $U_{12}^{\mathrm{T}}U_{12}=I_{n-r-s_1}$ and $U_{22}^{\mathrm{T}}U_{22}=I_{r-s_2}$, then (23) is equivalent to

$$\|G_1-U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}\|=\min_{G_1\in\mathrm{SR}^{(n-r-s_1)\times(n-r-s_1)}},\tag{24}$$

$$\|G_2-U_{22}^{\mathrm{T}}\widetilde{A}_{22}U_{22}\|=\min_{G_2\in\mathrm{SR}^{(r-s_2)\times(r-s_2)}}.\tag{25}$$

Since W_1 is an orthogonal matrix, and $U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}$ is a symmetric matrix, we have

$$\begin{aligned}&\|G_1-U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}\|^2\\&\quad=\left\|W_1\begin{pmatrix}\Sigma_1^{-1}Q_{11}^{\mathrm{T}}K_1Q_{11}\Sigma_1^{-1} & E_{12}\\ E_{12}^{\mathrm{T}} & E_{22}\end{pmatrix}W_1^{\mathrm{T}}-U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}\right\|^2\\&\quad=\left\|\Sigma_1^{-1}Q_{11}^{\mathrm{T}}K_1Q_{11}\Sigma_1^{-1}-W_{11}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{11}\right\|^2\\&\qquad+2\left\|E_{12}-W_{11}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{12}\right\|^2+\left\|E_{22}-W_{12}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{12}\right\|^2.\end{aligned}\tag{26}$$

Therefore (26) holds, which implies that (24) holds, if and only if $E_{12}=W_{11}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{12}$ and $E_{22}=W_{12}^{\mathrm{T}}U_{12}^{\mathrm{T}}\widetilde{A}_{11}U_{12}W_{12}$.

We can prove in a similar way that (25) holds if and only if $F_{12}=W_{21}^{\mathrm{T}}U_{22}^{\mathrm{T}}\widetilde{A}_{22}U_{22}W_{22}$ and $F_{22}=W_{22}^{\mathrm{T}}U_{22}^{\mathrm{T}}\widetilde{A}_{22}U_{22}W_{22}$.

Substituting E_{12}, E_{22} and F_{12}, F_{22} into (15), we get that the unique solution to Problem II can be expressed as (22) as desired. □

5. Conclusion

In this paper, we considered least squares solutions to the matrix equation $AX=B$ for bisymmetric matrices A under a central principal submatrix constraint and its optimal approximation. This least squares problem (Problem I) was presented in light of the special symmetric structure of bisymmetric matrices. Furthermore, it extended and developed the results derived in [3]. By applying the special properties, same structure and similar expression forms of bisymmetric matrices and their central principal submatrices, we converted Problem I to two least squares problems of half-sized independent symmetric matrices under themselves right-bottom principal submatrices constraint trickily. Then we derived some necessary and sufficient conditions for the solvability of Problem I, and obtained the representation of its general solutions. Moreover, we proved that Problem II (the optimal approximation) had a unique solution and gave its expression.

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